October Math Gems

Problem of the week 9

§1 Problems

Problem 1.1. When the letters of the alphabet are assigned to their integer values (A = 1, B = 2, C = 3, . . . , Z = 26), the word product for JULY is the product of the letter values for each of the letters in JULY. When the square root of JULY's word-product is put into its simplest radical form of $a\sqrt{b}$, where b has no perfect-square factors greater than 1, what is the value of b?

Solution. $\boxed{\textbf{Answer: 70}}$
The word product for JULY would be $10 \times 21 \times 12 \times 25$. Factorizing this expression, we get $2^3 \times 3^2 \times 5^3 \times 7$. Therefore, the value of b is $2 \times 5 \times 7 = 70$. □

Problem 1.2. In a certain triangle, the size of each of the angles is a whole number of degrees. Also, one angle is 30° larger than the average of the other two angles. What is the largest possible angle in this triangle?

Solution. $\boxed{\textbf{Answer: 99}}$
Let the average of the two angles, b and c for instance, be a. Therefore, $b + c = a$ and the third angle is $a + 30$. Therefore,

$$2a + a + 30 = 180$$

So, a is 50. This means that $b + c = 100°$ and the third angle is 80°. Now, the biggest angle possible is 99 so the other is equal to 1, which is the final answer. □

Problem 1.3. The points X, Y, and Z are the centers of three circles that touch externally, where each circle touches the other two. The triangle ABC has sides 13, 16, and 20. What are the radii of the three circles?

Solution. $\boxed{\textbf{Answer: 4.5, 8.5 and 11.5}}$
The sides are formed from the radii of the circles. Let the radii be a , b , and c. So,

$$a + b = 13$$

$$b + c = 16$$

$$c + a = 20$$

Solving this system of equations, we get that $a = 8.5$, $b = 4.5$, and $c = 11.5$ □

Problem 1.4. For which values of the positive integer n is it possible to divide the first $3n$ positive integers into three groups each of which has the same sum?

Solution. Let us split this question into two cases:

case 1: n is odd. We start with $n = 1$, which is impossible. Now, let us take $n = 3$. We can indeed split 1 to 9 into three groups, where each group has a sum of 15. Now, let us take $n + 2$, which will be odd. We can also split it into three groups of an equal sum. This means that any odd positive integer other than 1 has this property.

case 2: n is even. We start with $n = 2$. We can split 1 through 6 with groups that have a sum of 7. If we take $n + 2$ now, which will be even, we can similarly conclude it has the property.

Therefore, all positive integers n except 1 satisfy this property. □

Problem 1.5. what is the unit digit of $1! + 2! + 3! + 4! + ... + 2003!$?

Solution. $\boxed{\textbf{Answer: 3}}$
Since $n!$ is divisible by 10 for all $n \geq 5$, the units digit will be the units of $1!+2!+3!+4! = 33$. Therefore, the unit digit will be 3. □

Problem 1.6. A whole number between 1 and 99 is not greater than 90, not less than 30, not a perfect square, not even, not a prime, not divisible by 3, and its last digit is not 5. What is the number?

Solution. $\boxed{\textbf{Answer: 77}}$
From the question's first line, we are looking for a number between 30 and 90. From the second line, it is odd and not 49 or 81. Also it is not prime, so 31, 37, 41, 43, 47, 53, 57, 59, 61, 67, 71, 73, 79, 83, 87 and 89 are also excluded. Since it is not divisible by 3 or 5,$33, 35, 39, 45, 51, 55, 63, 65, 69, 75$ and 85 are also excluded.

The only remaining possibility is 77. □

Problem 1.7. How many triples (x, y, z) of positive integers satisfy $x^{yz} = 64$?

Solution. First, notice that $64 = 2^6$. So, x must be a power of 2 such that $x^{yz} = 64$. The only possibilities are $x = 2$, $yz = 6$ or $x = 4$, $yz = 3$, or $x = 8$, $yz = 2$ or $x = 64$, $yz = 1$. So altogether there are 9 solutions, $(x, y, z) = (2, 1, 6), (2, 2, 3), (2, 3, 2), (2, 6, 1), (4, 1, 3), (4, 3, 1),$ $(8, 1, 2), (8, 2, 1)$, and $(64, 1, 1)$. □

Problem 1.8. If $2x + 3y + z = 48$ and $4x + 3y + 2z = 69$, what is $6x + 3y + 3z$ equal to?

Solution. $\boxed{\textbf{Answer: 90}}$
First, notice that by adding the left sides and the right sides of the equations, you get $6x + 6y + 3z = 117$. We need to subtract $3y$ to get $6x + 3y + 3z$. But if you subtract $69 = 4x + 3y + 2z$ from $96 = 4x + 6y + 2z$, you get $3y = 27$. Hence $6x + 3y + 3z= 117 - 27 = 90$. □

Problem 1.9. The sum of six consecutive positive odd integers starting with n is a perfect cube. Find the smallest possible n.

Solution. $\boxed{\textbf{Answer: 31}}$
The sum of the six consecutive odd integers starting with $n = n + (n + 2) + (n + 4) + (n + 6) + (n + 8) + (n + 10) = 6n + 30 = 6(n + 5)$. The smallest cube of the form $6(n + 5)$ occurs when $n + 5 = 36$, so $n = 31$ □

Problem 1.10. Let a, b, and c be positive real numbers greater than or equal to 3. Prove that

$$3(abc + b + 2c) \geq 2(ab + 2ac + 3bc)$$

and determine all equality cases

Solution. Let $a = 3 + x$, $b = 3 + y$, and $c = 3 + z$, where x, y and z are ≥ 0. Now, we can write the expression to be

$$3xyz + 7xy + 5xz + 3yz + 9x + 6y + 3z \geq 0$$

which is obviously greater than 0. The equality is achieved only if $x = y = z = 0$, which is $a = b = c = 3$ □

Problem 1.11. Is there a triangle with an area of 12 cm^2 and a perimeter of 12 cm?

Solution. $\boxed{\textbf{Answer: No}}$
For a triangle with a fixed perimeter, its area is maximized when it is an equilateral triangle. So, its maximum area would be

$$\frac{\sqrt{3}}{4}s^2 = 4\sqrt{3} < 12$$

So, there is no such triangle. □

Problem 1.12. The real numbers a, b, c and d satisfy simultaneously the equations

$$abc - d = 1, \quad bcd - a = 2, \quad cda - b = 3, \quad dab - c = -6.$$

Prove that $a + b + c + d \neq 0$

Solution. Let us suppose that $a + b + c + d = 0$. Adding the original equations, we get

$$abc + bcd + cda + dab = 0$$

Substituting with $d = -a - b - c$, we get

$$-b^2c - bc^2 - a^2c - ac^2 - abc - a^2b - ab^2 - abc = 0$$

$$-(a + b)(b + c)(c + a) = 0$$

This means that either $a = -b$, $a = -c$, $a = -d$ Now,

$$a = -b \implies bcd + b = 2, -bcd - b = 3$$

$$a = -c \implies bcd + c = 2, -bcd - c = -6$$

$$a = -d \implies bcd - d = 2, -bcd - d = 1$$

All these expressions contradict our original assumption. So, $a + b + c + d \neq 0$ □

Problem 1.13. Let a, b and c be positive real numbers such that $abc = \frac{1}{8}$. Prove the inequality

$$a^2 + b^2 + c^2 + a^2b^2 + b^2c^2 + c^2a^2 \geq \frac{15}{16}$$

When does equality hold?

Solution. Let $a = \frac{x}{2y}, b = \frac{y}{2z}$ and $c = \frac{z}{2x}$. Therefore, we get

$$4x^4z^2 + 4y^4x^2 + 4z^4y^2 + x^4y^2 + z^4x^2 + y^4z^2 \geq 15x^2y^2z^2$$

This is true according to the AM-GM inequality. The equality happens when $a = b = c = \frac{1}{2}$ □

Problem 1.14. Solve for positive real numbers

$$n + \lfloor\sqrt{n}\rfloor + \lfloor\sqrt[3]{n}\rfloor = 2014$$

Solution. $\boxed{\textbf{Answer: 1958}}$
Let the expression $n + \lfloor\sqrt{n}\rfloor + \lfloor\sqrt[3]{n}\rfloor = f(n)$. $f(n)$ is increasing all its terms increases as n increases. You can also note that $\sqrt{n} + \sqrt[3]{n} \leq 56$ and observe the interval $1936 \leq n < 2014$. You will find that the solution is 1958 (Since the function is strictly increasing, there is only one solution). □

Problem 1.15. Let a be a positive real number such that $a^3 = 6(a+1)$. Prove that the equation $x^2 + ax + a^2 - 6$ has no real solutions.

Solution. For the equation to have no real solution, its discriminant must be less than 0. So, we can write it as

$$a^2 - 4(a^2 - 6) < 0$$

$$a > \sqrt{8}$$

However, this result doesn't satisfy the given equation for a. Therefore, the discriminant will be negative and the quadratic equation will have no real solution. □

Problem 1.16. Let $a, b, c \geq 0$ and $a + b + c = 2$. Prove that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{2c+1}{a+b} \geq \sqrt{10} - \frac{3}{4}$$

Solution. □

Problem 1.17. Let a, b be non-negative real numbers such that $a + b = 2$. Prove that

$$\frac{1}{a^2+1} + \frac{1}{b^2+1} \leq \frac{2}{ab+1}$$

Solution. LHS $= \frac{a}{2-a} + \frac{b}{2-b} + \frac{2c+1}{2-c} = \frac{2}{2-a} + \frac{2}{2-b} + \frac{5}{2-c} - 4 \geq \frac{(2\sqrt{2}+\sqrt{5})^2}{4} - 4 = \sqrt{10} - \frac{3}{4}$
(Cauchy-Schwartz inequality was used to go from step 2 to step 3) □

Problem 1.18. Let $x \leq 8$. Prove that

$$x(9-x) + \frac{16}{9-x} \leq 24$$

Solution. Let $a = 1+t$ and $b = 1-t$, where $-1 \leq t \leq 1$. Therefore, the original inequality can be written as

$$\frac{1}{t^2+2+2t} + \frac{1}{t^2+2-2t} \leq \frac{2}{2-t^2}$$

This is simplified to

$$\frac{2-t^4}{t^2+4} \leq 1$$

This obviously holds for all $-1 \leq t \leq 1$. Therefore, the original inequality is correct. □

Problem 1.19. Let a, b, c be positive real numbers such that $ab + bc + ca = 3$. Prove the following inequality

$$\frac{a}{a^2 - bc + 3} + \frac{b}{b^2 - ca + 3} + \frac{c}{c^2 - ab + 3} \leq \frac{3}{a + b + c}.$$

Solution. The given inequality could be written as

$$x(9 - x) + \frac{16}{9 - x} - 24 = \frac{(x - 8)(x - 5)^2}{9 - x} \leq 0$$

One can easily notice that this inequality would hold for any value of $x \leq 8$ $\square$

Problem 1.20. Let a, b be non-negative real numbers such that $a + b = 2$. Prove that

$$\frac{a^2 + a + 1}{b^2 - b + 3} + \frac{b^2 + b + 1}{a^2 - a + 3} \geq 2$$

Solution. For the first term on the LHS, we can rewrite it as

$$\frac{a}{a^2 - bc + 3} = \frac{a}{a^2 - bc + ac + ab + bc} = \frac{1}{a + b + c}$$

The same goes for the other two terms, and we get the sum on the LHS is equal to $\frac{3}{a+b+c}$. This proves the given inequality. $\square$